

Extra current and integer quantum Hall conductance in the spin-orbit coupling system

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Abstract. - We study the extra term of particle current in a 2D k -cubic Rashba spin-orbit coupling system and the integer quantization of Hall conductance in this system. We provide a correct formula of charge current in this system and the careful consideration of extra currents provides a stronger theoretical basis for the theory of quantum Hall effect which has not been considered before. The nontrivial extra contribution to particle current density and local conductivity, which originates from cubic dependence on the momentum operator in the Hamiltonian, will have no effect on the integer quantization of Hall conductance. The extension of Noether's theorem for the 2D k -cubic Rashba system is also addressed. Two methods reach to exactly the same results.

Introduction. – Recent experimental demonstrations of spin Hall effect in some semiconductors [1–5] may create a way to manipulate the spin of carriers in terms of electric field that presents potential in future applications. It has stimulated many scientists' interest. The experiments clearly show that the spin-orbit coupling (SOC) of carriers in some semiconductors plays a key role in disclosing the spin Hall effect. Several theoretical models of SOC have been suggested to study the charge and spin transport for different kinds of semiconductor systems, such as 2 dimensional (2D) linear k dependent Rashba [6] and Dresselhaus [7, 8] models, quadratic k dependent Luttinger model [9], 3D k -cubic Dresselhaus model [10] and 2D k -cubic Rashba SOC model which was found in a GaAs-Al_xGa_{1-x}As interface of a typical semiconductor heterojunction where the k -cubic Rashba SOC effect for heavy holes can not be neglected in a high-density regime [11]. For some Hamiltonians including terms with high order power (> 2) of momentum operators (MO) $\hat{\mathbf{p}}$, like 3D k -cubic Dresselhaus model, we have proved the conventional expression of particle current density (CD) $\mathbf{j}_{conv}(\mathbf{r}, t) = Re\{\psi^\dagger(\mathbf{r}, t)1/(i\hbar)[\mathbf{r}, \hat{H}]\psi(\mathbf{r}, t)\} = (1/e) Re\{\psi^\dagger(\mathbf{r}, t)(\partial\hat{H}/\partial\mathbf{A})\psi(\mathbf{r}, t)\}$ is no longer valid [12]. In that system, for the sake of current conservation, a nontrivial extra term of CD $\mathbf{j}_{extra}(\mathbf{r}, t)$ ($\nabla \cdot \mathbf{j}_{extra}(\mathbf{r}, t) \neq 0$) should be added to the conventional one.

Then the continuity equation of conserved particle CD $\mathbf{j}(\mathbf{r}, t) (= \mathbf{j}_{conv}(\mathbf{r}, t) + \mathbf{j}_{extra}(\mathbf{r}, t))$ can be satisfied. Thus, the extra term $\mathbf{j}_{extra}(\mathbf{r}, t)$ is a physical quantity and has effect on the conductivity of the system. It is natural to extend it to k -cubic Rashba system where the extra terms of CD may also appear due to cubic k , the term with high order power of MO $\hat{\mathbf{p}}$ in its Hamiltonian. However, 2D k -cubic Rashba is a real system that can demonstrate the integer quantum Hall conductance. The high precision of integer quantum Hall effect [13] was explained in some famous papers [14–16] where the expression of charge CD is implicitly based upon the conventional form. One naturally questions whether some correction to the quantum Hall conductance could come from the additional term of current $\mathbf{j}_{extra}(\mathbf{r}, t)$ in 2D k -cubic Rashba SOC semiconductors. In this paper, we would rigorously deduce the exact expression of charge CD that shows the existence of nontrivial extra term $\mathbf{j}_{extra}(\mathbf{r}, t)$ ($\nabla \cdot \mathbf{j}_{extra}(\mathbf{r}, t) \neq 0$). It is not a local circular current and does have the contribution to electric conductivity. Further, we prove that it has no contribution to the quantum Hall conductance. So the explanation of integer quantum Hall conductance is extended to a more general case that includes MO of triple power in the Hamiltonian, though whose formula of charge CD must be corrected by a nontrivial extra term due to the requirement of its continuity. Our paper shows a more clear understanding of the property of integer quantization of Hall conductance no matter the Hamiltonian including additional cubic k dependent SOC which is a realizable

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2D quantum Hall system.

This paper is organized as following. Firstly, we simply introduce the formulae of the calculation of particle CD in the first section. The deduction of the expression of extra term $\mathbf{j}_{extra}(\mathbf{r}, t)$ in addition to $\mathbf{j}_{conv}(\mathbf{r}, t)$ for a 2D cubic Rashba Hamiltonian is presented. The second section gives a proof that there is no contribution to the integer quantum Hall conductance from extra term $\mathbf{j}_{extra}(\mathbf{r}, t)$. Our expression of particle CD confirmed by extended Noether's theorem is attached in the appendix.

Density of particle current. – We study the 2D cubic Rashba system that is a promising model system for an ultra thin film of p -doped semiconductor [11]. In a perpendicular magnetic field, the single particle Hamiltonian is

$$\hat{H} = \hat{H}_N(\tilde{\mathbf{p}}, \mathbf{r}) + \hat{H}_R, \quad (1)$$

$$\hat{H}_N(\tilde{\mathbf{p}}, \mathbf{r}) = \tilde{\mathbf{p}}^2/2m^* + V(\mathbf{r}) - e\hat{y}E_y, \quad (2)$$

$$\hat{H}_R = i\lambda(\hat{p}_-^3\sigma^+ - \hat{p}_+^3\sigma^-) \quad (3)$$

where $V(\mathbf{r})$ is a local spin independent potential and could contain an impurity potential, $\lambda = \alpha/2\hbar^3$ is the spin-orbit coupling constant, E_y is the transverse Hall electric field, $\hat{p}_\pm = \hat{p}_x \pm i\hat{p}_y$, $\sigma^\pm = \sigma_x \pm i\sigma_y$ where σ_x and σ_y are Pauli matrices, and $\hat{\mathbf{p}} = (\hat{p}_x - eA_x, \hat{p}_y - eA_y)$. The corresponding Schrödinger (or say Pauli) equation is

$$\frac{\partial}{\partial t}\psi(\mathbf{r}, t) = \frac{1}{i\hbar}\hat{H}\psi(\mathbf{r}, t), \quad (4)$$

where Hamiltonian \hat{H} is a 2×2 matrix. The particle density for a pure quantum state is $n(\mathbf{r}, t) = \psi^\dagger(\mathbf{r}, t)\psi(\mathbf{r}, t)$ in which we have performed the inner product for spin space, but not for position. This rule of inner product is also used in the following deductions implicitly. Since the number of particles is conserved, the total number of particles $N = \int n(\mathbf{r}, t)d\mathbf{r}$ should be a constant. The conserved particle CD $\mathbf{j}(\mathbf{r}, t)$ is defined by the following continuity equation:

$$\frac{\partial n(\mathbf{r}, t)}{\partial t} = -\nabla \cdot \mathbf{j}(\mathbf{r}, t). \quad (5)$$

For simplifying the notations, in the paper, we will not discriminate the notions of particle CD and charge CD which only differ by a factor of charge e and can be self-explanatory according to the context.

For a mixed state, the density matrix $\hat{\rho} = \sum_n |\psi_n\rangle \rho_n \langle \psi_n|$ where ρ_n is the probability of the state $|\psi_n\rangle$, $\rho_n \geq 0$, $\sum_n \rho_n = 1$. The density of particle is defined by

$$\begin{aligned} n(\mathbf{r}, t) &= \langle \mathbf{r}, t | \hat{\rho} | \mathbf{r}, t \rangle = \sum_n \langle \mathbf{r}, t | \psi_n \rangle \rho_n \langle \psi_n | \mathbf{r}, t \rangle \\ &\equiv \sum_n \rho_n^\dagger \psi_n(\mathbf{r}, t) \psi_n(\mathbf{r}, t), \end{aligned} \quad (6)$$

We discuss the case of ρ_n being time independent. Then based on the Schrödinger equation, the left hand side of eq.(5) can be expressed as

$$\begin{aligned} \frac{\partial n(\mathbf{r}, t)}{\partial t} &= \sum_n \rho_n \{ -\nabla \cdot \mathbf{j}_N^n(\mathbf{r}, t) \\ &\quad + (\frac{1}{i\hbar} \hat{H}_R \psi_n(\mathbf{r}, t))^\dagger \psi_n(\mathbf{r}, t) \\ &\quad + \psi_n^\dagger(\mathbf{r}, t) (\frac{1}{i\hbar} \hat{H}_R \psi_n(\mathbf{r}, t)) \}, \end{aligned} \quad (7)$$

where $\mathbf{j}_N^n(\mathbf{r}, t) = \text{Re} \left\{ \psi_n^\dagger(\mathbf{r}, t) \left(\frac{1}{i\hbar} [\mathbf{r}, \hat{H}_N] \psi_n(\mathbf{r}, t) \right) \right\}$ is just the conventional expression of particle CD for non-SOC part $\hat{H}_N(\tilde{\mathbf{p}}, \mathbf{r})$. If the last two terms can be changed into $-\nabla \cdot \text{Re} \left\{ \psi_n^\dagger(\mathbf{r}, t) \left(\frac{1}{i\hbar} [\mathbf{r}, \hat{H}_R] \psi_n(\mathbf{r}, t) \right) \right\}$, the particle CD would be $\mathbf{j}^n(\mathbf{r}, t) = \text{Re} \left\{ \psi_n^\dagger(\mathbf{r}, t) \left(\frac{1}{i\hbar} [\mathbf{r}, \hat{H}] \psi_n(\mathbf{r}, t) \right) \right\}$, which is just the conventional formula $\mathbf{j}_{conv}^n(\mathbf{r}, t)$. However, it is not right. There should be an extra term $\mathbf{j}_{extra}(\mathbf{r}, t)$ in addition to the term $\mathbf{j}_{conv}^n(\mathbf{r}, t)$. And we will prove that $\nabla \cdot \mathbf{j}_{extra}(\mathbf{r}, t) \neq 0$. Thus, the conventional formula of particle CD is not conserved in 2D cubic Rashba system. For simplicity, we only consider the case in pure state and denote $\psi = \psi_n(\mathbf{r}, t)$. The expression of particle CD for mixed state can be easily obtained from the one of pure state. After some algebra, we obtain

$$\begin{aligned} &\left(\frac{1}{i\hbar} \hat{H}_R \psi \right)^\dagger \psi + \psi^\dagger \left(\frac{1}{i\hbar} \hat{H}_R \psi \right) \\ &= -\frac{1}{3} \nabla \cdot \left[\left(\frac{1}{i\hbar} [\mathbf{r}, \hat{H}_R] \psi \right)^\dagger \psi + \psi^\dagger \left(\frac{1}{i\hbar} [\mathbf{r}, \hat{H}_R] \psi \right) \right] \\ &\quad + \frac{1}{3} \left(\frac{1}{i\hbar} [\mathbf{r}, \hat{H}_R] \psi \right)^\dagger \cdot \nabla \psi + \frac{1}{3} (\nabla \psi)^\dagger \cdot \left(\frac{1}{i\hbar} [\mathbf{r}, \hat{H}_R] \psi \right) \\ &\quad - \frac{1}{3} \frac{eBy}{\hbar^2} \left\{ \left([x, \hat{H}_R] \psi \right)^\dagger \psi + \psi^\dagger [x, \hat{H}_R] \psi \right\} \\ &= -\nabla \cdot \mathbf{j}_R(\mathbf{r}, t). \end{aligned}$$

where

$$\begin{aligned} j_R^x(\mathbf{r}, t) &= \frac{1}{3} \left\{ \left(\frac{1}{i\hbar} [x, \hat{H}_R] \psi \right)^\dagger \psi + \psi^\dagger \left(\frac{1}{i\hbar} [x, \hat{H}_R] \psi \right) \right\} \\ &\quad + \lambda \text{Re} \left\{ (\partial_y \psi)^\dagger [x, \tilde{p}_-^2 \sigma^+ + \tilde{p}_+^2 \sigma^-] \psi \right\} \\ &\quad + \text{Re} \left\{ \frac{\lambda eBy}{\hbar} \psi^\dagger [x, \tilde{p}_-^2 \sigma^+ - \tilde{p}_+^2 \sigma^-] \psi \right\} \\ &\quad - 2\lambda \hbar^2 \left[(\partial_x \psi)^\dagger \sigma_y \partial_x \psi + (\partial_y \psi)^\dagger \sigma_y (\partial_y \psi) \right] \\ &\quad - i2\lambda \hbar eBy \left[(\partial_y \psi)^\dagger \sigma_x \psi - \psi^\dagger \sigma_x \partial_y \psi \right] \\ &\quad + 2\lambda (eB)^2 [\psi^\dagger y^2 \sigma_y \psi], \end{aligned} \quad (8)$$

$$\begin{aligned}
j_R^y(\mathbf{r}, t) = & \frac{1}{3} \left\{ \left(\frac{1}{i\hbar} [y, \hat{H}_R] \psi \right)^\dagger \psi + \psi^\dagger \left(\frac{1}{i\hbar} [y, \hat{H}_R] \psi \right) \right\} \\
& + \lambda Re \left\{ (\partial_y \psi)^\dagger [y, \tilde{p}_-^2 \sigma^+ + \tilde{p}_+^2 \sigma^-] \psi \right\} \\
& + Re \left\{ \frac{\lambda e B y}{\hbar} \psi^\dagger [y, \tilde{p}_-^2 \sigma^+ - \tilde{p}_+^2 \sigma^-] \psi \right\} \\
& + 2\lambda \hbar^2 \left[(\partial_x \psi)^\dagger \sigma_x \partial_x \psi + (\partial_y \psi)^\dagger \sigma_x \partial_y \psi \right] \\
& - i 2\lambda \hbar e B y \left[(\partial_y \psi)^\dagger \sigma_y \psi - \psi^\dagger \sigma_y \partial_y \psi \right] \\
& - 2\lambda (eB)^2 [\psi^\dagger y^2 \sigma_x \psi]. \tag{9}
\end{aligned}$$

In the above, following relations are applied:

$$\begin{aligned}
\hat{H}_R &= -\frac{1}{3} \nabla \cdot [\mathbf{r}, \hat{H}_R] + \frac{eBy}{3} \frac{1}{i\hbar} [x, \hat{H}_R], \\
\tilde{p}_-^2 \sigma^+ + \tilde{p}_+^2 \sigma^- &= -\frac{1}{2} \nabla \cdot [\mathbf{r}, \tilde{p}_-^2 \sigma^+ + \tilde{p}_+^2 \sigma^-] \\
&\quad + \frac{1}{2} eBy \frac{1}{i\hbar} [x, \tilde{p}_-^2 \sigma^+ + \tilde{p}_+^2 \sigma^-].
\end{aligned}$$

So we have $\mathbf{j}(\mathbf{r}, t) = \mathbf{j}_N(\mathbf{r}, t) + \mathbf{j}_R(\mathbf{r}, t)$. Then the extra term is

$$\begin{aligned}
\mathbf{j}_{extra}(\mathbf{r}, t) &= \mathbf{j}(\mathbf{r}, t) - \mathbf{j}_{conv}(\mathbf{r}, t) \\
&= \mathbf{j}_R(\mathbf{r}, t) - Re \left\{ \psi^\dagger \frac{1}{i\hbar} [\mathbf{r}, \hat{H}_R] \psi \right\},
\end{aligned}$$

where $\mathbf{j}_{conv}(\mathbf{r}, t) = Re\{\psi^\dagger 1/(i\hbar)[\mathbf{r}, \hat{H}]\psi\}$ is the so called conventional current that widely appeared in literatures. Then the expression of extra particle CD can be finally simplified as

$$j_{extra}^x(\mathbf{r}, t) = 2\lambda \hbar^2 \partial_x \partial_y (\psi^\dagger \sigma_x \psi) - \lambda \hbar^2 \partial_x^2 (\psi^\dagger \sigma_y \psi) + \lambda \hbar^2 \partial_y^2 (\psi^\dagger \sigma_y \psi), \tag{10}$$

$$j_{extra}^y(\mathbf{r}, t) = 2\lambda \hbar^2 \partial_x \partial_y (\psi^\dagger \sigma_y \psi) + \lambda \hbar^2 \partial_x^2 (\psi^\dagger \sigma_x \psi) - \lambda \hbar^2 \partial_y^2 (\psi^\dagger \sigma_x \psi). \tag{11}$$

In above equations, all $(\psi^\dagger \sigma_\alpha \psi) = (\psi^\dagger(\mathbf{r}, t), \sigma_\alpha \psi(\mathbf{r}, t))$ are position dependent. The divergence of $\mathbf{j}_{extra}(\mathbf{r}, t)$ is generally non-zero, $\nabla \cdot \mathbf{j}_{extra}(\mathbf{r}, t) = -\lambda \hbar^2 \partial_x^3 (\psi^\dagger \sigma_y \psi) - \lambda \hbar^2 \partial_y^3 (\psi^\dagger \sigma_x \psi) + 3\hbar^2 \lambda \partial_x \partial_y^2 (\psi^\dagger \sigma_y \psi) + 3\lambda \hbar^2 \partial_x^2 \partial_y (\psi^\dagger \sigma_x \psi) \neq 0$. So, as shown in Eqs.(10) and (11), we derived non-trivial extra terms in the expression of conserved particle CD of a 2D cubic Rashba Hamiltonian. The same result of extra currents can also be obtained by taking account of the gauge invariance based on Noether's theorem. Its detail is presented in appendix.

When we consider a mixed state, the extra term of charge CD can be expressed as

$$\mathbf{j}_{extra}(\mathbf{r}, t) = e \sum_n \rho_n \mathbf{j}_{extra}^{(n)}(\mathbf{r}, t). \tag{12}$$

In fact, our Hamiltonian $H(= H_N + H_R)$ is time independent, so the particle density n and charge CD \mathbf{j} are position dependent only.

Hall conductance. – Now we study the charge CD along x direction which is $j^{(x)}(\mathbf{r}) = j_{conv}^{(x)}(\mathbf{r}) + j_{extra}^{(x)}(\mathbf{r})$. We take the integral with respect to y for $j^{(x)}(\mathbf{r})$ to get the charge current

$$I^{(x)} = \int_{L_y} j_{conv}^{(x)}(\mathbf{r}) dy + \int_{L_y} j_{extra}^{(x)}(\mathbf{r}) dy, \tag{13}$$

where L_y is the width of the system. And denote the length of the system as L_x . Finally, L_x and L_y can approach to infinity if the system becomes macroscopic. $I^{(x)}$ should not be position x dependent because of the particle conservation. Then we take the integral of x for $I^{(x)}$:

$$\begin{aligned}
I^{(x)} &= \frac{1}{L_x} \int_{L_x} \int_{L_y} j_{conv}^{(x)}(\mathbf{r}) d\mathbf{r} + \frac{1}{L_x} \int_{L_x} \int_{L_y} j_{extra}^{(x)}(\mathbf{r}) d\mathbf{r} \\
&= \frac{1}{\Omega} \iint_{\Omega} j_{conv}^{(x)}(\mathbf{r}) L_y d\mathbf{r} + \frac{1}{\Omega} \iint_{\Omega} j_{extra}^{(x)}(\mathbf{r}) L_y d\mathbf{r} \\
&= I_{conv}^{(x)} + I_{extra}^{(x)}. \tag{14}
\end{aligned}$$

where $\Omega = L_x L_y$, $I_{conv}^{(x)} = \frac{1}{\Omega} \iint_{\Omega} j_{conv}^{(x)}(\mathbf{r}) L_y d\mathbf{r}$ and $I_{extra}^{(x)} = \frac{1}{\Omega} \iint_{\Omega} j_{extra}^{(x)}(\mathbf{r}) L_y d\mathbf{r}$. From Eq.(10), the extra part of the current is

$$\begin{aligned}
I_{extra}^{(x)} &= e \lambda \hbar^2 L_y \sum_n \rho_n \left\{ \frac{1}{\Omega} \iint_{\Omega} [2 \partial_x \partial_y (\Psi_n^\dagger(\mathbf{r}) \sigma_x \Psi_n(\mathbf{r})) \right. \\
&\quad \left. - \partial_x^2 (\Psi_n^\dagger(\mathbf{r}) \sigma_y \Psi_n(\mathbf{r})) + \partial_y^2 (\Psi_n^\dagger(\mathbf{r}) \sigma_y \Psi_n(\mathbf{r}))] d\mathbf{r} \right\} \\
&= -e \lambda L_y \sum_n \rho_n \frac{1}{\Omega} \iint_{\Omega} d\mathbf{r} \{ 2 (\hat{p}_x \hat{p}_y \Psi_n(\mathbf{r}))^\dagger \sigma_x \Psi_n(\mathbf{r}) \\
&\quad + 2 \Psi_n^\dagger(\mathbf{r}) \sigma_x \hat{p}_x \hat{p}_y \Psi_n(\mathbf{r}) - 2 (\hat{p}_x \Psi_n(\mathbf{r}))^\dagger \sigma_x \hat{p}_y \Psi_n(\mathbf{r}) \\
&\quad - 2 (\hat{p}_y \Psi_n(\mathbf{r}))^\dagger \sigma_x \hat{p}_x \Psi_n(\mathbf{r}) - (\hat{p}_x^2 \Psi_n(\mathbf{r}))^\dagger \sigma_y \Psi_n(\mathbf{r}) \\
&\quad - \Psi_n^\dagger(\mathbf{r}) \sigma_y \hat{p}_x^2 \Psi_n(\mathbf{r}) + 2 (\hat{p}_x \Psi_n(\mathbf{r}))^\dagger \sigma_y \hat{p}_x \Psi_n(\mathbf{r}) \\
&\quad + (\hat{p}_y^2 \Psi_n(\mathbf{r}))^\dagger \sigma_y \Psi_n(\mathbf{r}) + \Psi_n^\dagger(\mathbf{r}) \sigma_y \hat{p}_y^2 \Psi_n(\mathbf{r}) \\
&\quad \left. - 2 (\hat{p}_y \Psi_n(\mathbf{r}))^\dagger \sigma_y \hat{p}_y \Psi_n(\mathbf{r}) \right\}. \tag{15}
\end{aligned}$$

The terms in right side of the above equation become the spatial inner product after the integration of \mathbf{r} over the whole space of the system. Since the operators $\{\hat{p}_x, \hat{p}_y\}$ are hermitian, as an example, we have

$$\begin{aligned}
&\frac{1}{\Omega} \iint_{\Omega} d\mathbf{r} \{ 2 (\hat{p}_x \hat{p}_y \Psi_n(\mathbf{r}))^\dagger \sigma_x \Psi_n(\mathbf{r}) \} \\
&= \frac{1}{\Omega} \iint_{\Omega} d\mathbf{r} \{ 2 (\hat{p}_y \Psi_n(\mathbf{r}))^\dagger \sigma_x \hat{p}_x \Psi_n(\mathbf{r}) \} \\
&= \frac{1}{\Omega} \iint_{\Omega} d\mathbf{r} \{ 2 (\hat{p}_x \Psi_n(\mathbf{r}))^\dagger \sigma_x \hat{p}_y \Psi_n(\mathbf{r}) \} \\
&= \frac{1}{\Omega} \iint_{\Omega} d\mathbf{r} \{ 2 (\Psi_n(\mathbf{r}))^\dagger \sigma_x \hat{p}_x \hat{p}_y \Psi_n(\mathbf{r}) \}.
\end{aligned}$$

Considering above property for inner product of position space in equation (15), we can easily obtain $I_{extra}^{(x)} = 0$. No contribution to Hall conductance from extra term of charge CD is proved. Finally, we have

$$\begin{aligned}
I^{(x)} &= L_y \frac{1}{\Omega} \iint_{\Omega} j_{conv}^{(x)}(\mathbf{r}) d\mathbf{r} \\
&= eL_y Re \sum_n \rho_n \frac{1}{\Omega} \iint_{\Omega} \left\{ \Psi_n^\dagger(\mathbf{r}) \frac{1}{i\hbar} [\hat{x}, \hat{H}] \Psi_n(\mathbf{r}) \right\} d\mathbf{r} \\
&= eL_y Re \sum_n \rho_n \langle \Psi_n | \frac{1}{i\hbar} [\hat{x}, \hat{H}_0] | \Psi_n \rangle, \\
H_0 &= \hat{p}^2 / (2m^*) + i\lambda(\hat{p}_-^3 \sigma^+ - \hat{p}_+^3 \sigma^-)
\end{aligned}$$

Thus, the quantum Hall conductance is only from the conventional term $\mathbf{j}_{conv}(\mathbf{r}, t)$. For H_0 , the cubic 2D Rashba model without transverse electric field, its Schrödinger equation is

$$H_0 |\Psi_n^{(0)}\rangle = E_n^{(0)} |\Psi_n^{(0)}\rangle.$$

It has been solved exactly [17],

$$\begin{aligned}
E_n^{(0)} &= (n + 1/2)\hbar\omega, n \leq 2, \\
E_{n,s}^{(0)} &= \left[(n-1) + s\sqrt{\gamma^2 n(n-1)(n-2) + \frac{9}{4}} \right] \hbar\omega, \\
n &\geq 3.
\end{aligned} \tag{16}$$

where $\omega = eB/(mc)$, $s = \pm 1$ and $\gamma = 4\lambda m^* \sqrt{2\hbar e B/c}$. The eigen energies in traditional quantum Hall effect are Landau levels separated by gaps. Now the “Landau levels” of a 2D cubic Rashba model have some modification for $n \geq 3$, but they still keep the essential feature of the gap separation. The corresponding eigenfunctions are

$$\begin{aligned}
|\Psi_n^{(0)}\rangle &= \begin{pmatrix} 0 \\ \phi_n \end{pmatrix}, n \leq 2, \\
|\Psi_n^{(0)}\rangle &= |\Psi_{n,s}^{(0)}\rangle = \begin{pmatrix} C_{ns1}\phi_{n-3} \\ C_{ns2}\phi_n \end{pmatrix}, n \geq 3.
\end{aligned} \tag{17}$$

where $\{C_{ns1}, C_{ns2}\}$ are normalized constants,

$$\begin{aligned}
C_{ns1} &= \frac{ic_{n,s}}{\sqrt{c_{n,s}^2 + 1}}, C_{ns2} = \frac{1}{\sqrt{c_{n,s}^2 + 1}}, \\
c_{n,s} &\equiv \frac{1}{\gamma\sqrt{n(n-1)(n-2)}} \\
&\times \left(-\frac{3}{2} + s\sqrt{\gamma^2 n(n-1)(n-2) + \frac{9}{4}} \right),
\end{aligned}$$

and ϕ_n is the wave function of harmonic oscillation type. Impurities may result in widening out the “Landau levels”. The conventional velocity operator in position space is $\hat{\mathbf{v}} = 1/(i\hbar)[\mathbf{r}, H] = 1/(i\hbar)[\mathbf{r}, H_0]$. It is easy to have $\hat{\mathbf{v}}(\mathbf{k}) = 1/\hbar \nabla_{\mathbf{k}} E^{(0)}(\mathbf{k})$ in \mathbf{k} space. Following Laughlin [14] or Kohmoto’s [16] deduction, the integer quantization of

quantum Hall conductance can be obtained. Here we will present a different approach to reach to the conclusion of integer quantum Hall conductance for such a specific 2D cubic Rashba system.

Since the eigen energies and wave functions of Schrödinger equation in the second quantization representation can be found exactly, we also calculate the Hall conductance in linear response approximation and it shows excellent consistency with integer quantization of Hall conductance. More specifically, the Hall conductance σ_{xy} of this system can be written as $\sigma_{xy} = \sum_{n,s} N_{n,s} (\sigma_{xy})_{n,s}$, where $N_{n,s}$ is the number of particles occupying the (n, s) -th “Landau level” (here the “Landau level” is marked by two index, n indicating the energy level of the system without SOC, s indicating the energy level splitting due to SOC). And $(\sigma_{xy})_{n,s}$ is the one particle’s contribution from the (n, s) -th “Landau level”, by linear response theory,

$$\begin{aligned}
(\sigma_{xy})_{n,s} &= \sum_{(n'', s'') \neq (n, s)} \\
&\times \left[\frac{\langle \Psi_{n,s}^{(0)} | \hat{j}_x | \Psi_{n'', s''}^{(0)} \rangle \langle \Psi_{n'', s''}^{(0)} | H' | \Psi_{n,s}^{(0)} \rangle}{E_y L_x L_y (E_{n,s}^{(0)} - E_{n'', s''}^{(0)})} + h.c. \right] \tag{18}
\end{aligned}$$

where $H' = -e\hat{y}E_y$ and the electric field is uniform. Now we adopt the Landau gauge $\hat{p}_x = \hbar k_x - eBy/c$, $\hat{p}_y = \hat{p}_y$, and introduce the operator of bosonic quasi particles $a = \sqrt{\frac{c}{2\hbar e B}} (\hat{p}_x - i\hat{p}_y)$, $a^\dagger = \sqrt{\frac{c}{2\hbar e B}} (\hat{p}_x + i\hat{p}_y)$, $[a, a^\dagger] = 1$. Then we get

$$\hat{H}_0 = \hbar\omega \begin{pmatrix} a^\dagger a + \frac{1}{2} & i\gamma a^3 \\ -i\gamma a^{\dagger 3} & a^\dagger a + \frac{1}{2} \end{pmatrix}.$$

Then

$$\begin{aligned}
\hat{j}_x &= \frac{e}{i\hbar} [x, \hat{H}_0] \\
&= e\omega \sqrt{\frac{\hbar c}{2eB}} (a + a^\dagger) + \frac{3ie\gamma}{m} \sqrt{\frac{\hbar e B}{2c}} \begin{pmatrix} 0 & a^2 \\ -a^{\dagger 2} & 0 \end{pmatrix} \tag{19}
\end{aligned}$$

And

$$\hat{H}' = -e\hat{y}E_y = -eE_y \left(\hbar k_x - \sqrt{\frac{\hbar c}{2eB}} (a + a^\dagger) \right) \cdot I, \tag{20}$$

where I is a unit matrix. Then using Eqs.(17), (19) and (20) for matrix elements $\langle \Psi_{n,\pm}^{(0)} | \hat{j}_x | \Psi_{n+1,\pm}^{(0)} \rangle$ and $\langle \Psi_{n,\pm}^{(0)} | H' | \Psi_{n+1,\pm}^{(0)} \rangle$, the selection rules will be found. And the summation over states in Hall conductance in Eq.(18)

can be simplified as

$$\begin{aligned}
& (\sigma_{xy})_{n,\pm} \\
= & \frac{1}{EL_x L_y} \left[\frac{\langle \Psi_{n,\pm}^{(0)} | \hat{j}_x | \Psi_{n+1,+}^{(0)} \rangle \langle \Psi_{n+1,+}^{(0)} | H' | \Psi_{n,\pm}^{(0)} \rangle}{E_{n,\pm}^{(0)} - E_{n+1,+}^{(0)}} \right. \\
& + \frac{\langle \Psi_{n,\pm}^{(0)} | \hat{j}_x | \Psi_{n-1,+}^{(0)} \rangle \langle \Psi_{n-1,+}^{(0)} | H' | \Psi_{n,\pm}^{(0)} \rangle}{E_{n,\pm}^{(0)} - E_{n-1,+}^{(0)}} \\
& + \frac{\langle \Psi_{n,\pm}^{(0)} | \hat{j}_x | \Psi_{n+1,-}^{(0)} \rangle \langle \Psi_{n+1,-}^{(0)} | H' | \Psi_{n,\pm}^{(0)} \rangle}{E_{n,\pm}^{(0)} - E_{n+1,-}^{(0)}} \\
& \left. + \frac{\langle \Psi_{n,\pm}^{(0)} | \hat{j}_x | \Psi_{n-1,-}^{(0)} \rangle \langle \Psi_{n-1,-}^{(0)} | H' | \Psi_{n,\pm}^{(0)} \rangle}{E_{n,\pm}^{(0)} - E_{n-1,-}^{(0)}} + h.c. \right]
\end{aligned}$$

Then using eqs.(16) and (17), all the elements $\langle \Psi_{n,\pm}^{(0)} | \hat{j}_x | \Psi_{n\pm 1,\pm}^{(0)} \rangle$ and $\langle \Psi_{n\pm 1,\pm}^{(0)} | H' | \Psi_{n,\pm}^{(0)} \rangle$ can be calculated without difficulty. After long but straight algebraic deduction, we can finally obtain

$$(\sigma_{xy})_{\tilde{n},\pm} = -\frac{e^2}{h} \frac{\Phi_0}{\Phi}, \quad (21)$$

where $\Phi_0 = BL_x L_y$, $\Phi = hc/e$. By summing up all the contributions from different energy levels, the total Hall conductance will be

$$\begin{aligned}
\sigma_{xy} &= \sum_{\tilde{n}} \left((\sigma_{xy})_{\tilde{n},+} + (\sigma_{xy})_{\tilde{n},-} \right) = -\sum_{\tilde{n},s} N_{\tilde{n},s} \frac{e^2}{h} \frac{1}{\Phi/\Phi_0} \\
&= -\frac{e^2}{h} \frac{N_0}{\Phi/\Phi_0} = -\nu \frac{e^2}{h}.
\end{aligned} \quad (22)$$

Here N_0 is the total number of carriers, and the filling factor $\nu = N_0 / (\Phi/\Phi_0)$.

Due to the existence of impurities in practical samples, localized states appear in the region between “Landau levels”. It leads to the appearance of the plateaus when the Fermi level lies in that region. The gap between two conductance plateaus is obviously e^2/h . It is concluded that the cubic SOC do induce the extra term of CCD that yields the contribution to electric conductivity, but no contribution is given to the quantum Hall conductance.

Conclusions. — We have derived an exact formula of particle current density for a 2D cubic Rashba model that appears in some p -doped semiconductors. In addition to the conventional current expression, there must be an extra term that ensures the current continuity equation. The extra term must have the contribution to electric conductivity, but no contribution to the charge quantum Hall conductance that is proved rigorously. So, it can be clearly shown that no effect is made on the topological property of integer quantization of Hall conductance due to the existence of extra terms in the 2D cubic Rashba coupling system. Further experimentally detectable effects of the new term are still on research.

Appendix: Deduction of extra terms from Noether’s theorem. — In this appendix, we point out that, for a 2D cubic Rashba Hamiltonian where the highest order of derivatives is higher than 2, it is necessary to generalize the expression of conserved current in Noether’s theorem. Applying the generalized Noether’s theorem [12], we can get the expressions of conserved particle CD of a k -cubic Rashba SOC system from $U(1)$ gauge invariance.

Noether’s theorem, not only indicates the relation between conserved currents and symmetries of Lagrangian, but also implies that the expression of conserved current depends on the form of Lagrangian from the beginning of its deduction. In usual cases, Lagrangians are expressed as $\mathcal{L}[\phi(x), \partial_\mu \phi(x), \phi^\dagger(x), \partial_\mu \phi^\dagger(x)], x^\mu = (t, \mathbf{r}), \mu = 0, 1, 2, 3$ -such as the Lagrangian of complex scalar field- which only include fields $\phi(x), \phi^\dagger(x)$ and their first order derivatives $\partial_\mu \phi(x), \partial_\mu \phi^\dagger(x)$ as independent variables. But in our case, Hamiltonian \hat{H} includes higher order derivatives. So its Lagrangian should be written in the form $\mathcal{L}[\phi(x), \partial_\mu \phi(x), \partial_\mu \partial_\nu \phi(x), \dots, \phi^\dagger(x), \partial_\mu \phi^\dagger(x), \partial_\mu \partial_\nu \phi^\dagger(x), \dots]$, where higher-order derivatives are also included as independent variables. For simplicity, we denote $\phi(x)$ and $\phi^\dagger(x)$ as ϕ and ϕ^\dagger . The Hamiltonian of a k -cubic Rashba system studied here is $\hat{H}_R = \hat{p}^2 / (2m) + i\lambda (\hat{p}_-^2 \sigma^+ - \hat{p}_+^2 \sigma^-)$. The corresponding Lagrangian can be

$$\begin{aligned}
& \mathcal{L}[\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, \partial_\mu^2 \partial_\nu \phi; \phi^\dagger, \partial_\mu \phi^\dagger, \partial_\mu \partial_\nu \phi^\dagger, \partial_\mu^2 \partial_\nu \phi^\dagger] \\
&= \phi^\dagger (i\partial_0 \phi) + \frac{1}{2m} \phi^\dagger (\partial_x^2 + \partial_y^2) \phi \\
&+ 2i\lambda \phi^\dagger (\sigma^x \partial_y^3 + \sigma^y \partial_x^3) \phi \\
&- 2i\lambda \phi^\dagger (3\sigma^x \partial_x^2 \partial_y + 3\sigma^y \partial_x \partial_y^2) \phi.
\end{aligned}$$

According to the least action principle, one can easily obtain an Euler-Lagrange equation

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \phi)} - \partial_\mu^2 \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu^2 \partial_\nu \phi)},$$

which yields the Schrödinger equation. Actually, the first two terms on the right-hand side of the above equation give the conventional formula of particle CD. The remaining parts lead to the extra terms. The corresponding conserved current F_μ is

$$\begin{aligned}
F^\mu &= \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \phi)} \delta (\partial_\nu \phi) - (\partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \phi)}) \delta \phi \\
&+ \frac{\partial \mathcal{L}}{\partial (\partial_\mu^2 \partial_\nu \phi)} \delta (\partial_\mu \partial_\nu \phi) - (\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu^2 \partial_\nu \phi)}) \delta (\partial_\nu \phi) \\
&+ (\partial_\nu^2 \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu^2 \phi)}) \delta \phi + (\phi \rightarrow \phi^*),
\end{aligned} \quad (A.1)$$

which satisfies the continuity equation $\partial_\mu F^\mu = 0$. We concentrate on the deduction of conserved particle cur-

rent corresponding to $U(1)$ gauge symmetry. From infinitesimal variation of fields $\delta\phi = i\alpha\phi$, $\delta\phi^\dagger = -i\alpha\phi^\dagger$, the expression of conserved particle CD for a k -cubic Rashba system is

$$\begin{aligned}\mathbf{j}^x &= -F^x = \phi^\dagger \left(\frac{i\partial_x}{2m} \phi \right) + \left(\frac{i\partial_x}{2m} \phi \right)^\dagger \phi \\ &\quad - 2\lambda[\phi^\dagger \sigma^y (\partial_x^2 \phi) + (\partial_x^2 \phi^\dagger) \sigma^y \phi \\ &\quad - (\partial_x \phi^\dagger) \sigma^y (\partial_x \phi)] + 6\lambda[\phi^\dagger (\sigma^x \partial_x \partial_y \phi) \\ &\quad - (\partial_x \phi^\dagger) (\sigma^x \partial_y \phi) + (\partial_y^2 \phi^\dagger) (\sigma^y \phi)], \\ \mathbf{j}^y &= -F^y = \phi^\dagger \left(\frac{i\partial_y}{2m} \phi \right) + \left(\frac{i\partial_y}{2m} \phi \right)^\dagger \phi \\ &\quad - 2\lambda[\phi^\dagger \sigma^x (\partial_y^2 \phi) + (\partial_y^2 \phi^\dagger) \sigma^x \phi \\ &\quad - (\partial_y \phi^\dagger) \sigma^x (\partial_y \phi)] + 6\lambda[\phi^\dagger (\sigma^y \partial_x \partial_y \phi) \\ &\quad - (\partial_y \phi^\dagger) (\sigma^y \partial_x \phi) + (\partial_x^2 \phi^\dagger) (\sigma^x \phi)].\end{aligned}$$

Comparing the above formulae with the conventional one $\mathbf{j}_{conv} = \text{Re} \left\{ \phi^\dagger \left(\frac{1}{i} [\mathbf{r}, H_R] \phi \right) \right\}$, we get the extra term of particle CD $\mathbf{j}_{extra} = \mathbf{j} - \mathbf{j}_{conv}$:

$$\begin{aligned}\mathbf{j}_{extra}^x &= -\lambda \partial_x^2 (\phi^\dagger \sigma^y \phi) + 6\lambda (\partial_x \phi^\dagger) \sigma^x (\partial_y \phi) \\ &\quad + 6\lambda (\partial_x \partial_y \phi^\dagger) (\sigma^x \phi) - 6\lambda (\partial_y^2 \phi^\dagger) (\sigma^y \phi) \\ &\quad + 3\lambda \phi^\dagger (\sigma^y \partial_y^2 \phi) + 3\lambda (\partial_y^2 \phi^\dagger) (\sigma^y \phi), \quad (\text{A.2})\end{aligned}$$

$$\begin{aligned}\mathbf{j}_{extra}^y &= -\lambda \partial_y^2 (\phi^\dagger \sigma^x \phi) + 6\lambda (\partial_y \phi^\dagger) \sigma^y (\partial_x \phi) \\ &\quad + 6\lambda (\partial_x \partial_y \phi^\dagger) (\sigma^y \phi) - 6\lambda (\partial_x^2 \phi^\dagger) (\sigma^x \phi) \\ &\quad + 3\lambda \phi^\dagger (\sigma^x \partial_x^2 \phi) + 3\lambda (\partial_x^2 \phi^\dagger) (\sigma^x \phi). \quad (\text{A.3})\end{aligned}$$

Further, it is not difficult to check that the extra term \mathbf{j}_{extra} deduced here by extended Noether's theorem and \mathbf{j}_{extra} in the second section do satisfy the equation $\nabla \cdot (\mathbf{j}_{extra} - \mathbf{j}_{extra}) = 0$. Thus we conclude that our result of extra term is rigorous.

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